2.1.1

Prop: $G(x_b, x_a, \omega) = \sum_n \frac{\psi_n(x_b)\,\psi_n^+(x_a)}{\omega - \epsilon_n}$

Proof: Starting from the definition

$$G(x_b, x_a, t) = -i\Theta(t)\langle x_b | U(t) | x_a\rangle$$

and $G(x_b, x_a, \omega) = \int_{-\infty}^{\infty} dt\, e^{+it\omega} G(x_b, x_a, t)$

We may insert a resolution of the identity $\mathrm{id} = \sum_n |n\rangle\langle n|$, while noting that $U(t)|n\rangle = e^{-i\epsilon_n t}|n\rangle$.

Hence

Note have commuted $\Sigma, \int$

$$G(x_b, x_a, \omega)$$

$$= -i \sum_n \int_0^{\infty} dt\, e^{it(\omega - \epsilon_n)} \langle x_b | n\rangle\langle n | x_a\rangle$$

$$= \sum_n \frac{\psi_n^+(x_b)\,\psi(x_a)}{\omega - \epsilon_n + i0^+}$$

What is the pole structure of $G(0,0,\omega)$ for SHO?

For SHO

$$G(0,0,t) = -i\left(\frac{m\omega_0}{2\pi i \sin t\omega_0}\right)^{1/2}$$
$$= -i(m\omega_0/\pi)^{1/2}\left\{e^{it\omega_0} - e^{-it\omega_0}\right\}^{-1/2}$$
$$= -i\,(m\omega_0/\pi)^{1/2}\, e^{-t\omega_0/2}\sum_{n=0}^{\infty} e^{-in\omega_0 t}$$

So $$G(0,0,\omega) = (m\omega_0/\pi)^{1/2}\sum_{n=0}^{\infty}\frac{1}{\omega - \omega_0(n+1/2)}.$$

Hence $G(0,0,\omega)$ has simple poles at $\omega = \omega_0(n+1/2)$ for $n=0,1,\ldots$ with residue $(m\omega_0/\pi)^{1/2} = |\psi_n(0)|^2$ as expected from $H = \omega_0(a^+a+1/2)$

2.1.2 For a free particle the propagator is clearly

$$G(k_b, k_a, \omega) = \int \frac{d^n k}{(2\pi)^n} \frac{\langle k_b | k \rangle \langle k | k_a \rangle}{\omega - k^2/2m + i0^+}$$

$$= \frac{(2\pi)^n \, \delta^n(k_b - k_a)}{\omega - k_a^2/2m + i\,0^+}$$

2.1.3

A coherent state $|\alpha\rangle$ is defined by $a|\alpha\rangle = \alpha|\alpha\rangle$

It follows that

Resolution of identity: $\quad \mathrm{id} = \int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha|$

Inner prod: $\quad \langle\alpha|\beta\rangle = e^{-\{|\alpha|^2+|\beta|^2\}/2} e^{\alpha^+\beta}$

From this, the coherent state propagator is

$$iG(\alpha_b t_b|\alpha_a t_a)$$

$$= \Theta(t) \langle\alpha_b| U(t_b, t_a)|\alpha_a\rangle$$

$$= \int \prod_{i=1}^{N-1} \left[\frac{d^2\alpha_i}{\pi} \langle\alpha_{i+1}| U(t_{i+1}, t_i)|\alpha_i\rangle\right]$$

$$= \int \prod_{i=1}^{N-1} \left[\frac{d^2\alpha_i}{\pi} \langle\alpha_{i+1}| e^{-i\Delta t\,\omega_0 a^+ a}|\alpha_i\rangle\right]$$

with $\alpha_0 = \alpha_a$
$\alpha_N = \alpha_b$
$t_i = t_a + i\Delta t$
$\Delta t = (t_b - t_a)/N$

$$= \int \prod_{i=1}^{N-1} \left[\frac{d^2\alpha_i}{\pi} e^{-i\Delta t \omega \alpha_{i+1}^+ \alpha_i - \frac{1}{2}\left(|\alpha_i|^2 + |\alpha_{i+1}|^2\right) + \alpha_{i+1}^+ \alpha_i} \right]$$

$$= \int \frac{1}{\pi^{N-1}} D_N^2\alpha \, e^{iS_N} \quad \text{with } D_N^2\alpha = \prod_{i=1}^{N-1} d^2\alpha_i$$

and

$$S_N = \sum_{i=1}^{N} \Delta t \left\{ -\omega \alpha_{i+1}^+ \alpha_i - i\left(\frac{\alpha_{i+1}^+ - \alpha_i^+}{2\Delta t}\right)\alpha_i + i\,\alpha_{i+1}^+ \left(\frac{\alpha_{i+1} - \alpha_i}{2\Delta t}\right) \right\}$$

as $N \to \infty$

$$S_N \to S = \int_{t_a}^{t_b} dt \left\{ \frac{i}{2}\left(\alpha^+\dot{\alpha} - \dot{\alpha}^+\alpha\right) - \omega\alpha^+\alpha \right\}$$

$$= \int_{t_a}^{t_b} dt \, L$$

The corresponding classical Lagrangian is

$L_c = p\dot{x} - H$

and

$$L - L_c = \frac{i}{4}\frac{d}{dt}\left(\alpha^2 + \alpha^{+2}\right).$$

is simply a total derivative, which is undetectable

2.1.4 Routine excersize to obtain imag time prop

$G(x_b, x_a, \tau)$ w. $\Delta\tau = \tau/N$

$$:= \langle x_b | e^{-\tau H} | x_a \rangle$$

$$= \int \frac{dx_1 \cdots dx_{N-1}\, dp_1 \cdots dp_N}{(2\pi)^N} \langle x_b | e^{-\tau H/N} | p_N \rangle \langle p_N | x_{N-1} \rangle \langle x_{N-1} | \cdots | p_1 \rangle \langle p_1 | x_a \rangle$$

$$= \int \mathcal{D}_N x\, \mathcal{D}_N p\; e^{-\sum_{i=1}^{N} \left\{ V(x_i) + \frac{p_i^2}{2m} \right\} \Delta\tau}\; e^{i \sum_{i=1}^{N} p_i (x_i - x_{i-1})}$$

$$\overset{N \to \infty}{=} \int \mathcal{D}x, p\; e^{-\int_0^\tau d\tau' \left\{ \frac{p^2}{2m} + V(x) - ip\dot{x} \right\}}$$

w. $\mathcal{D}_N x, p = \prod_{i=1}^{N-1} dx_i \prod_{i=1}^{N} \frac{dp_i}{2\pi}$

Can evaluate as Gaussian using

$$-\frac{p^2}{2m} + ip\dot{x} = -\frac{(p - i\dot{x}m)^2}{2m} - \frac{m}{2}\dot{x}^2$$

so $p - im\dot{x} \rightsquigarrow \tilde{p}$ substitution gives

$$G(x_b, x_a, \tau)$$

$$= \int D_N x \left(\frac{m}{2\pi\Delta\tau}\right)^{\sim} e^{-\int_0^\tau d\tau' \left\{\frac{m\dot{x}^2}{2} + V(x)\right\}}$$

$$D_N x = \prod_{i=1}^{N-1} dx_i$$

2.1.5

Prop: The propagator od a free particle in constant dorce f is

$$iG(x_b, t \mid x_a, 0)$$
$$= \left(\frac{m}{2\pi i t}\right)^{1/2} \exp\left\{ i \left(\frac{m(x_b - x_a)^2}{2t} + \frac{ft}{2}(x_a + x_b) - \frac{f^2 t^3}{24m} \right) \right\}$$

Prood: using that $S[d] = S'[0] + \int dt\, f\, x(t)$ and S is quadratic

$$iG = A^{\sim} \int \mathcal{D}x \; e^{iS}$$

$$S' = \int dt \, \frac{m\dot{x}^2}{2} + f x$$

$$x = x_{cl} + \delta x \qquad \frac{\delta S'}{\delta x_{cl}} = 0$$

$$\hookrightarrow \quad iG = \left(\frac{m}{2\pi i t}\right)^{1/2} e^{i S'_{cl}}$$

Solving for classical action

$$\ddot{x}_{cl} = f/m \qquad x(0) = x_a$$
$$x(t) = x_b$$

$$x_{cl}(t') = \frac{f}{2m} t'^2 + x_a + v_0 t'$$

$$x_{cl}(t) = \frac{f}{2m} t^2 + x_a + v_0 t$$
$$= x_b$$

$$v_0 = \frac{x_b - x_a}{t} - \frac{f t}{2m}$$

So

$$S_{cl} = \int_0^t dt' \, \frac{m}{2}\left(\frac{f}{m}t' + v_0\right)^2 + f\,x(t')$$

$$= \frac{f^2 t^3}{m}\left(\frac{1}{6} + \frac{1}{6} - \frac{1}{4} + \frac{1}{8} - \frac{1}{4}\right)$$

$$+ \frac{ft}{2}\Big(2x_a + x_b - x_a$$

$$+ x_b - x_a - (x_b - x_a)\Big)$$

$$+ m(x_b - x_a)^2/2t$$

$$\hookrightarrow$$

$$iG = \sqrt{\frac{m}{2\pi i t}}\, e^{i\left\{\frac{m}{2t}(x_b - x_a)^2 + \frac{1}{2}ft(x_a + x_b) - f^2 t^3/24m\right\}}$$

2.1.6 What is prop of SHO?

S is quadratic

$\hookrightarrow iG = A(t)\, e^{iS_{cl}}$

So need to find S_{cl}:

EOM: $\ddot{x} + \omega_0^2 x = 0$

$\hookrightarrow \quad x = A e^{i\omega_0 t} + B e^{-i\omega_0 t}$

With bound.

Conditions:

$$A + B = x_a$$

$$A e^{i\omega_0 t} + B e^{-i\omega_0 t} = x_b$$

$$\begin{pmatrix} x_a \\ x_b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ z & z^{-1} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

where $z = e^{i\omega_0 t}$.

As such

$$S_{cl} = \int \frac{1}{2}\left(m\dot{x}_{cl}^2 - m\omega_0^2 x_{cl}^2\right) dt$$

$$= \int \frac{d}{dt}\frac{m\dot{x}x}{2} - \frac{1}{2} m x_{cl} \underbrace{\left(\ddot{x}_{cl} + \omega_0^2 x_{cl}\right)}_{0} dt$$

$$= \frac{m}{2}\left[x\dot{x}\right]_0^t$$

$$\dot{x}(t) = i\omega\left[Ae^{i\omega t} - Be^{-i\omega t}\right]$$

$$x(0)\,\dot{x}(0) = i\omega\left[A^2 - B^2\right]$$

$$x(t)\,\dot{x}(t) = i\omega\left[A^2 z^2 - B^2/z^2\right]$$

Hence

$$S_{cl} = \frac{im\omega}{2}\left\{\underbrace{A^2(z^2-1) + B^2(1-z^{-2})}_{X}\right\}$$

Now

$$\begin{pmatrix} x_a \\ x_b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ z & z^{-1} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \underbrace{\frac{1}{z^{-1}-z} \begin{pmatrix} z^{-1} & -1 \\ -z & 1 \end{pmatrix}}_{M} \begin{pmatrix} x_a \\ x_b \end{pmatrix}$$

$$X = \underline{M}^T \begin{pmatrix} z^2-1 & 0 \\ 0 & 1-z^{-2} \end{pmatrix} \underline{M}$$

$$= \frac{1}{(z-z^{-1})^2} \begin{pmatrix} z^{-1} & -z \\ -1 & 1 \end{pmatrix} \begin{pmatrix} z-z^{-1} & 1-z^2 \\ z^{-1}-z & 1-z^{-2} \end{pmatrix}$$

$$= \frac{1}{(z-z^{+})^2} \begin{pmatrix} -z^{-2}+z^2 & z^{-1}-z-z+z^{-1} \\ 2(z^{-1}-z) & z^2-z^{-2} \end{pmatrix}$$

so

$$S_{cl} = \frac{im\omega_0}{2(z-z^{-1})} \begin{pmatrix} x_a \\ x_b \end{pmatrix}^{+} \begin{pmatrix} z+z^{-1} & -2 \\ -2 & z+z^{-1} \end{pmatrix} \begin{pmatrix} x_a \\ x_b \end{pmatrix}$$

$$= \frac{m\omega_0}{2\sin(\omega t)} \left\{ \cos\omega t \,(x_a^2 + x_b^2) - 2x_a x_b \right\}$$

Now $A(t)$ from normalisation

$$\delta(x) = \int_{\mathbb{R}} G(x_b t; 0,0)\, G^*(x_b t; x, 0)\, dx_b$$

$$= |A|^2 \int_{\mathbb{R}} dx_b \exp\left\{ \frac{im\omega_0}{2\sin t\omega_0} \left[2x x_b - x^2 \right] \right\}$$

integrating over x

$$1 = |A|^2 \int dx\, dx_b\, e^{-i\frac{B(t)}{2}[x^2 - 2xx_b]}$$

$$= |A|^2 \int dx\, dx_b\, e^{-i\frac{B(t)}{2}(x-x_b)^2 + \frac{iB}{2}x_b^2}$$

$$= |A|^2 \left| \frac{2\pi \sin(t\omega_0)i}{m\omega_0} \right|$$

$$\hookrightarrow A = \left(\frac{m\omega_0}{2\pi i \sin t\omega_0}\right)^{1/2} e^{i\phi(t)}$$

2.1.7 Obtain real t from imag t

$$G = A_\tau^N \int D_N x \, e^{-\int_0^\tau d\tau' \frac{m\dot{x}^2}{2}}$$

$$\left.\begin{array}{l} x_a = x(0) \\ x_b \doteq x(\tau) \end{array}\right\} \Rightarrow x_{cl} = x_a + \frac{x_b - x_a}{\tau}\tau'$$

so $S_{cl} = \frac{m}{2\tau}(x_b - x_a)^2$

Now fluctuation contribution is

$$\int D\delta x \, e^{-\int_0^\tau d\tau' m \frac{\delta \dot{x}^2}{2}} \qquad \delta x\Big|_{0,\tau} = 0$$

$$= \left(\frac{m}{2\pi\tau}\right)^{1/2}$$

Hence

$$\hookrightarrow G = \left(\frac{m}{2\pi\tau}\right)^{1/2} e^{-\frac{m}{2\tau}(x_b - x_a)^2}$$

analytic continuation

$$\tau \to t\, e^{i\vartheta}$$

$$G = \left(\frac{m}{2\pi t e^{i\vartheta}}\right)^{1/2} e^{-\frac{m(x_b - x_a)^2}{2t} e^{-i\vartheta}}$$

well defined about $\vartheta = 0$

eg take branch cut of

$\sqrt{}$ at $(-\infty, 0)$

$$\vartheta \to \pi$$

$$G = \left(\frac{m}{2\pi t i}\right)^{1/2} e^{\frac{im(x_b - x_a)^2}{2t}}$$

2.1.8 $$S_{SHO} = \frac{m}{2}\int_0^{\tau} d\tau' \, x\left[-\frac{d^2}{d\tau'^2} + \omega_0^2\right]x$$

$$G(0,0,\tau)$$

$$= \int \mathcal{D}x \; e^{-S_{SHO}}$$

$$= \left(\frac{2\pi}{m \det\left(-d^2/d\tau'^2 + \omega_0^2\right)}\right)^{1/2}$$

$$\left[-\frac{d^2}{d\tau'^2} + \omega_0^2\right] y_N = \lambda_N y_N$$

$$\omega^2 + \omega_0^2 = \lambda_N$$

$$\omega = \frac{\pi N}{\tau} \qquad N = 0, \pm 1, \ldots$$

ratio ω ω_0 $(\omega_0 \to 0)$

$$\frac{\omega_0}{\omega_0}\prod_N 1 + \left(\frac{\omega \tau}{\pi N}\right)^2 = \frac{\sinh(\omega_0 \tau)}{\tau \omega_0}$$

↳ and Now Compare w. free

$$\hookrightarrow \quad G(0,0,\tau) = \left\{ \frac{m\omega_0}{2\pi \sinh(\omega_0 \tau)} \right\}^{1/2}$$

As $\tau \to \infty$

$$G(0,0,\tau) \to \left(\frac{m\omega}{\pi} \right)^{1/2} e^{-\frac{\omega_0}{2}\tau}$$

$$= |\psi_0(0)|^2 e^{-E_0 \tau}$$

where $E_0 = \frac{\omega_0}{2}$:) as expected